Study on the Strong Coupling Coexistence of the Mutualism Population Model with Dirichlet Boundary Conditions

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Abstract
Biological mathematics is a cross subject. In the biological field, the mathematical model of the object is established, the mathematical model of the research object is established, and the model is qualitatively and quantitatively analyzed, so as to explain the biological phenomenon, which is a hot spot in the research of modern applied mathematics. In this paper, the single population, multi population and Lgoistic classical biological population models were studied. A mathematical model was established to describe the process of the two populations of mutual competition, mutual dependence, and the law of the jungle.

The stability of equilibrium state was studied using the stability theory of differential equation, and the conditions to produce this result were analyzed. The reciprocal model with Dirichlet boundary conditions was studied, and the general case strongly coupled elliptic problem solutions were introduced. By using the method of upper and lower solutions, the Schauder fixed point theorem and the monotone iterative method; the conditions for the existence of the strong coupling problem of the reciprocity model were presented. It was found that when the cross diffusion and inter population effect were relatively weak, there was at least one positive solution to the strong coupling system.

Keywords: Dirichlet boundary condition, mutual symbiosis, biological population, strong coupling

Introduction

The systematic study of ecological mathematics began in 1920s. American Ecological mathematician Lotka in 1921 studied the chemical reaction and the Italy mathematician Volterra in 1923 studied the fish competition, and they, respectively, proposed the classic Lotka-Volterra model.

In the traditional sense, mathematics, biology, physics and other disciplines have been recognized as a separate subject. But from 1950s, due to the extensive use of machinery industry and information technology, mutual penetration and integration between disciplines appeared, cross discipline was more and more obvious, and boundaries between many disciplines were not particularly clear[1]. Mathematics and computer science had been developed as a tool for many disciplines or fields, science and engineering research could not be separated from mathematics and computer science and technology. As a new field of research, the development of biological mathematics could not be separated from the auxiliary function of mathematics and computer [2]. In recent years, the reaction diffusion equation of ecological model has attracted the attention of many experts.

Both biological and mathematical researchers have done research of it, so that the development of biology has been into a new era, and the research of biological mathematics has achieved a new breakthrough [3]. The mathematical biology was that using mathematical theory and computer technology to study on the quantitative and spatial structure of life sciences, analyze the intrinsic properties of complex biological systems, and reveal the biological information in a large number of biological experiments[4].

Biological mathematics includes many branches, such as biological statistics, mathematical biology, quantitative genetics, and so on. As one of the branches, population biology has been widely studied by scholars at home and abroad [5]. As the first persons of population biology, Lotka and Volterra developed the classic Lotka-Volterra model, which included predator-prey model, competition model and cooperative model [6]. For the first two kinds of models, domestic and foreign experts and scholars have made a wealth of research results. Capitán J proposed a density dependent population model to observe the phenomenon of population isolation[7].

For the classical biological competition model with cross diffusion, the existence of positive solutions had been studied. But for a more general system of two equations, the two equations were density dependent, and the existence of positive solutions was also proved using the fixed-point method by Bronstein J L[8].

Gan and Lin studied a class of Lotka-Volterra model with strong coupling and competition between three populations. Georgelin E studied the stability of the solution of a class of two population model with diffusion. In his paper, he considered such a model, and gave the existence and uniqueness of its solution [9]. In Qi Ji's research, he studied a diffusive predator prey model of two species, which not only gave the uniqueness and coexistence of the solution, but also studied its asymptotic properties[10].

Compared with the prey and competition system, the study of reciprocity system the same importance is relatively less. In this paper, a class of two population reciprocity model is described, that is, the emergence of a population has a good effect on the other populations. The upper and lower solutions and the monotone iterative method are used to study the coexistence of the two-species reciprocal model, and the conditions of coexistence are given.

Biological population model

Mathematical model of single population: Any species in nature is not isolated, but closely related to other populations in the biological community. Single population model is to consider only one kind of biological community, without considering the effect of other population factors on it. In ecology, the single population model is the simplest and most basic model, and the research on them is the basis for the study of more complex models [11].

For populations with long life, overlapping generations, and a large number, the change of quantity can be regarded as a continuous process and can be described by the following differential equation.

$$\frac{dN(t)}{dt} = f(t, N(t)) \quad (1)$$

For populations with relatively short life and non-overlapping generations, or populations with long life, overlapping generations, and relatively small number, the number of changes can be described by a general difference equation model.

$$N_{t+1} = f(t, N(t)) \quad (2)$$

Among them, $N(t)$ is the population quantity of time t, which is the function of time t. In addition to including t and $N(t)$, f often also contains the intrinsic population growth rate r and environmental capacity K.

Malthus model: $\frac{dN}{dt} = rN$, N is the population quantity of time t, and r is the intrinsic population growth rate. There is $N(t) = N(t_0)e^{r(t-t_0)}$.

Logistic model: $\frac{dN}{dt} = r(1-\frac{N}{K})N$, K represents the maximum capacity of the population. There is $N(t) = \frac{K}{1+\frac{K-N(t_0)}{N(t_0)}e^{-r(t-t_0)}}$.

General population model: $\frac{dN}{dt} = Nf(N)$.

$\frac{dN}{dt} = Nf(N) - h$ is with constant harvesting rate.

$\frac{dN}{dt} = Nf(N) - h(t)$ is with time varying harvesting rate.

Two population model: Assuming x (t) and y (t) represents, respectively, the number and density of two populations in the time t, and for the establishment of models, their relative growth rate $\frac{1}{x}\frac{dx}{dt}, \frac{1}{y}\frac{dy}{dt}$ should be considered. Taking into account the two aspects of the development law and the influence of the interaction between the two sides, it is commonly used in the form of:

$$\frac{1}{x}\frac{dx}{dt} = f_1(x,y) \quad \frac{dx}{dt} = x(a_1 + b_1 x + c_1 y)$$
$$\frac{1}{y}\frac{dy}{dt} = f_2(x,y), \quad \frac{dy}{dt} = y(a_2 + b_2 x + c_2 y) \quad (3)$$

Among them, a_1 and a_2 are, respectively, the natural growth rate of population x and y, and their positive and negative are determined by their respective food sources. For example, when the food of the x population is natural resources out of the y population, $a_1 \geq 0$; and when the food of the x population is only from organisms of the y population, $a_1 \leq 0$. $b_1 x$ and $c_2 y$ are the reflections of the density control factor inside a variety of groups, that is, the kind of internal competition, so $b_1 \leq 0, c_2 \leq 0$.

The positive and negative of b_2 and c_1 should be decided according to the interaction between these two populations, which is generally divided into the following three kinds of situations.

Competing type: Two populations, or kill each other, or compete with the same kind of food resources, each of which has a disadvantage to each other, so: $c_1 \leq 0, b_2 \leq 0$.
Mutually beneficial coexistence: The existence of two populations is beneficial to each other, and can promote the growth of each other, then $c_1 \geq 0, b_2 \geq 0$.

Predator and prey type: The y population uses the x population as food source. At this time, the presence of x population is beneficial to the growth of y population, and y is not favorable for x, so $c_1 \leq 0, b_2 \geq 0$.

Logistic classical model: The basic assumption of the famous Malthus population growth model is that the growth rate of population r is constant, or that the growth of population in the unit time is proportional to the population quantity at that time. The population growth curve of this model is J - shaped. But when the population is small (relative to resources), the population growth can be approximated as a constant. When the population quantity increases to a certain degree, the growth rate will gradually decrease with the increase of population quantity, that is, the density control leads to the decrease of r with the increase of density. At this time the population growth will no longer be J - shaped, but S - shaped. There are two characteristics of the S curve: the curve is asymptotic to the K value, namely the equilibrium density; the curve is smooth [12]. The simplest mathematical model of generating S curve is to add a density control factor $\left(1-\frac{N}{K}\right)$ on the exponential growth equation, namely Malthus model, and the famous Logistic equation can be obtained:

$$\frac{dN(t)}{dt}=rN(t)\left(1-\frac{N(t)}{K}\right) \tag{4}$$

There is an analytical solution for the Logistic model and a method for solving the model is given:

This is a variable separation equation, and after the separation of variables, it can be obtained:

$$\frac{dN}{N\left(1-\frac{N}{K}\right)}=rdt \tag{5}$$

From 0 to t on both sides, for the definite integrals:

$$\int_{N_0}^{N(t)}\frac{dN}{N(1-N/K)}=\int_0^t rdt=rt \tag{6}$$

The left side of the equation is calculated for the definite integrals, and it can be obtained.

$$\int_{N_0}^{N(t)}\left(\frac{1}{N}+\frac{1/K}{1-N/K}\right)=\left[\ln(N)-\ln(1-N/K)\right]_{N_0}^{N(t)}$$
$$=\ln(N(t))-\ln(1-N(t)/K)-\ln(N_0)+\ln(1-N_0/K) \tag{7}$$

And:

$$\ln\left(\frac{N}{1-N/K}\bullet\frac{1-N_0/K}{N_0}\right)=rt \tag{8}$$

The analytical solution is:

$$N(t)=\frac{KN_0}{N_0-(N_0-K)e^{-rt}} \tag{9}$$

Competition, dependence, and law of the jungle model of biological population

Competition model of populations: When the two populations compete for the same food source and living space, the common result is that the population with weak competitiveness is extinct, and the population with the competitive ability can achieve the maximum capacity of the environment. The mathematical model is established to describe the process of the competition between the two populations and to analyze the conditions of this result [13]. Note: x1 (t), and x2 (t) are, respectively, the number of two populations of a and b; r1and r2 are, respectively, the natural growth rate of two populations of a and b; N1andN2 are, respectively, the maximum capacity of two populations of a and b.

First of all, the model assumes that there are two populations of a and b, and the number of their own survival time are subject to Logistic rule;

$$\dot{x}_1(t)=r_1x_1(1-\frac{x_1}{N_1}),\quad \dot{x}_2(t)=r_2x_2(1-\frac{x_2}{N_2}) \tag{10}$$

Here, the factor $1-x_1/N_1$ reflects the blocking effect of A consumption of limited resources on the growth of its own, and x_1/N_1 can be understood as the amount of food to support A when the number is x_1 relative to the N_1.

When two populations live in the same natural environment, taking into account the effects of B consumption of the same resource on the growth of A, we subtract one item from the factor $1-x_1/N_1$. The item is proportional to the x_2 number of population B, thus, for the A population:

$$\dot{x}_1(t)=r_1x_1\left(1-\frac{x_1}{N_1}-\sigma_1\frac{x_2}{N_2}\right),$$
$$\dot{x}_2(t)=r_2x_2\left(1-\sigma_2\frac{x_1}{N_1}-\frac{x_2}{N_2}\right) \tag{11}$$

For the A consumption of resource, B (relative to N_2) is the σ_1 times (relative to N_1) of B. When $\sigma_1>1$, for the blocking effect of the growth of A, B is greater than A and the competitiveness of B is bigger.

Model analysis: when $t\to\infty$, the trend of $x_1(t)$ and $x_2(t)$, namely the equilibrium point and its stability.

The equilibrium point and stability of $\begin{matrix}\dot{x}_1(t)=f(x_1,x_2)\\ \dot{x}_2(t)=g(x_1,x_2)\end{matrix}$: the equilibrium point $P_0\left(x_1^0,x_2^0\right)$ is the solution to the algebraic equation of $\begin{matrix}f(x_1,x_2)=0\\ g(x_1,x_2)=0\end{matrix}$.

If we start from any initial value of a neighborhood of P_0, there is $\lim_{t\to\infty}x_1(t)=x_1^0$ and $\lim_{t\to\infty}x_2(t)=x_2^0$. And P_0 is the stable equilibrium point of the differential equation.

P1 and P2 are the equilibrium points for a species to survive and the other to extinct, and P3 is the equilibrium point of the coexistence of the two species. The conditions for the stability of P1 is s1<1.

Phase trajectory analysis of the stability of equilibrium point:

$$\dot{x}_1(t) = r_1 x_1 \left(1 - \frac{x_1}{N_1} - \sigma_1 \frac{x_2}{N_2}\right)$$
$$\dot{x}_2(t) = r_2 x_2 \left(1 - \sigma_2 \frac{x_1}{N_1} - \frac{x_2}{N_2}\right)$$

$$\phi(x_1, x_2) = 1 - \frac{x_1}{N_1} - \sigma_1 \frac{x_2}{N_2}$$
$$\psi(x_1, x_2) = 1 - \sigma_2 \frac{x_1}{N_1} - \frac{x_2}{N_2} \qquad (12)$$

The phase trajectories starting from any point (t=0) tend to P_1 (N_1, 0) ($t\to\infty$), and P_1 (N_1, 0) is the stable equilibrium point. Conditions of stability of P1: the direct method s2>1, coupled with the s1<1 different from the (4) phase region, and the global stability can be inferred.

Table 1
Population competition equilibrium and stability of the model

Balance	p	q	Stability condition
$p_1(N_1, 0)$	$r_1 - r_2(1-\sigma_2)$	$-r_1 r_2(1-\sigma_2)$	$\sigma_2 > 1, \sigma_1 < 1$
$p_2(0, N_2)$	$-r_1(1-\sigma_1) + r_2$	$-r_1 r_2(1-\sigma_1)$	$\sigma_1 > 1, \sigma_2 < 1$
$p_3\left(\frac{N_1(1-\sigma_1)}{1-\sigma_1\sigma_2}, \frac{N_2(1-\sigma_2)}{1-\sigma_1\sigma_2}\right)$	$\frac{r_1(1-\sigma_1) + r_2(1-\sigma_2)}{1-\sigma_1\sigma_2}$	$\frac{r_1 r_2(1-\sigma_1)(1-\sigma_2)}{1-\sigma_1\sigma_2}$	$\sigma_1 < 1, \sigma_2 < 1$
$p_4(0,0)$	$-(r_1 + r_2)$	$r_1 r_2$	unstable

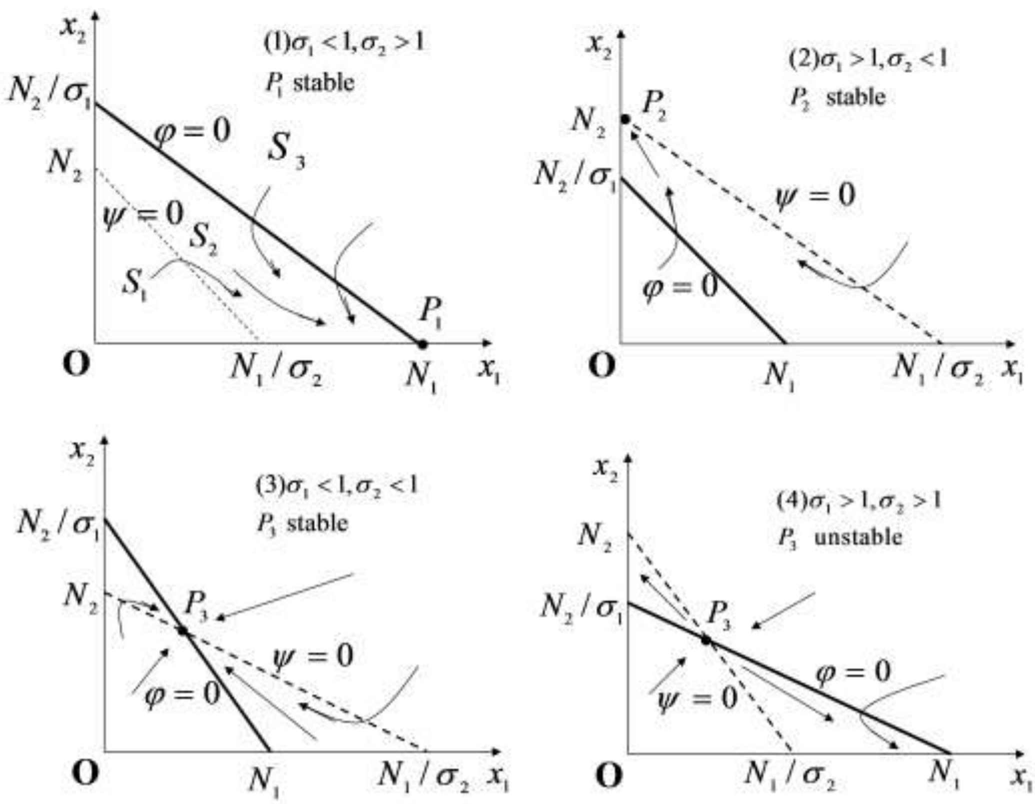

Fig. 1: Stability of phase trajectory

It can be seen from the figure above, the stable conditions of P_1: $\sigma_1 < 1, \sigma_2 > 1$. For the consumption of A resource, B (relative to N2) is σ_1 times (relative to N1) of A. Because $\sigma_1 < 1$, for the blocking effect of the growth of A, B is less than A, and the competitiveness of B is smaller. When $\sigma_2 > 1$, the competitiveness of A is bigger, and when A reaches the maximum capacity, the B will be extinct. The stable conditions of P_2: $\sigma_1 > 1, \sigma_2 < 1$, and the stable conditions of P3: $\sigma_1 < 1, \sigma_2 < 1$. But in general $\sigma_1 \approx 1/\sigma_2$, so it cannot meet the stable conditions of P3 [14].

Interdependence model of populations: There are a lot of two populations in the same environment in nature, such as plants and insects, and insects can help plants for fully pollination, and insects are also getting food (pollen). There is a similar relationship between humans and animals [15]. The mutual dependence of the two populations of A and B has three forms. In the first kind, A can survive alone, B cannot survive alone; A and B can survive with each other to provide food, and promote growth. In the second kind, A and B can survive alone; A and B can survive with each other to provide food, and promote growth. In the third kind, A and B cannot survive alone; A and B can survive with each other to provide food, and promote growth.

Model assumptions: A can survive alone, the number of changes in compliance with the laws of Logistic; When A and B together survive, B provides food for A and promotes growth of A [16]. B can't live alone; When A and B together survive, A provides food for B, and promotes the growth of B; The growth of B is also affected by its blocking effect (subject to Logistic rules).

$\dot{x}_1(t_1) = r_1 x_1 \left(1 - \frac{x_1}{N_1} + \sigma_1 \frac{x_2}{N_2}\right)$ means that the food provided by B is the σ_1 times of the food consumption of A.

$\dot{x}_2(t) = r_2 x_2 \left(-1 + \sigma_2 \frac{x_1}{N_1} - \frac{x_2}{N_2}\right)$ means that the food provided by A is the σ_2 times of the food consumption of B.

P_2 is the symbiotic equilibrium point of the interdependent of A and B, and the phase trajectory of stability of P_2:

$P_2\left(\frac{N_1(1-\sigma_1)}{1-\sigma_1\sigma_2}, \frac{N_2(\sigma_2-1)}{1-\sigma_1\sigma_2}\right)$, $\sigma_1<1$, $\sigma_2>1$, $\sigma_1\sigma_2<1$, P_2 is in stable.

$$\dot{x}_1(t_1) = r_1 x_1 \left(1 - \frac{x_1}{N_1} + \sigma_1 \frac{x_2}{N_2}\right) = r_1 x_1 \phi(x_1, x_2)$$
$$\dot{x}_2(t) = r_2 x_2 \left(-1 + \sigma_2 \frac{x_1}{N_1} - \frac{x_2}{N_2}\right) = r_2 x_2 \psi(x_1, x_2) \tag{13}$$

In fig. 2,
$S_1: \dot{x}_1 > 0, \dot{x}_2 < 0; S_2: \dot{x}_1 > 0, \dot{x}_2 > 0; S_3: \dot{x}_1 < 0, \dot{x}_2 > 0; S_4: \dot{x}_1 < 0, \dot{x}_2 < 0$.
It can be seen that P_2 is in stable.

Result analysis: A can survive alone, $\dot{x}_1(t_1) = r_1 x_1 \left(1 - \frac{x_1}{N_1} + \sigma_1 \frac{x_2}{N_2}\right)$, B cannot survive alone, $\dot{x}_2(t) = r_2 x_2 \left(-1 + \sigma_2 \frac{x_1}{N_1} - \frac{x_2}{N_2}\right)$.

In the $P_2\left(\frac{N_1(1-\sigma_1)}{1-\sigma_1\sigma_2}, \frac{N_2(\sigma_2-1)}{1-\sigma_1\sigma_2}\right)$, the stability condition of P2 is $\sigma_1<1$, $\sigma_2>1$, $\sigma_1\sigma_2<1$.
And, when $\sigma_2>1$, A must provide adequate food for B, and the food provided by A is the σ_2 times of the food consumption of B. $\sigma_1\sigma_2<1$ and $\sigma_2>1$ are the necessary conditions for the existence of P2. And σ_1 must small enough, so that $\sigma_1\sigma_2<1$ under the condition of $\sigma_2>1$.

Table 2
Populations dependent equilibrium and stability of the model

Balance	**p**	**q**	**Stability condition**
$p_1(N_1, 0)$	$r_1 - r_2(\sigma_2 - 1)$	$-r_1 r_2(\sigma_2 - 1)$	$\sigma_2 < 1, \sigma_1\sigma_2 < 1$
$P_2\left(\frac{N_1(1-\sigma_1)}{1-\sigma_1\sigma_2}, \frac{N_2(\sigma_2-1)}{1-\sigma_1\sigma_2}\right)$	$\frac{r_1(1-\sigma_1) + r_2(\sigma_2-1)}{1-\sigma_1\sigma_2}$	$\frac{r_1 r_2(1-\sigma_1)(\sigma_2-1)}{1-\sigma_1\sigma_2}$	$\sigma_1 < 1, \sigma_2 > 1,$ $\sigma_1\sigma_2 < 1$
$P_3(0, 0)$	$-r_1 + r_2$	$-r_1 r_2$	unstable

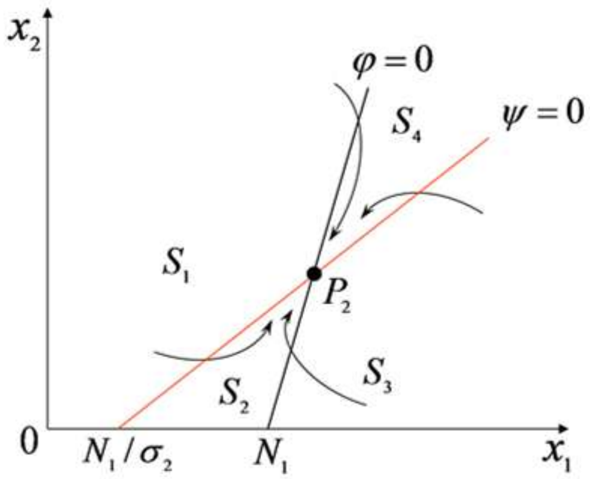

Fig. 2: P_2 stability of equilibrium phase trajectory

Law of the jungle model of population: Population A depends on abundant natural resources to survive and population B depends on the predation of armor for a living, thus forming predator-prey system, such as eating fish and sharks, leopard and lynx, pests and insects.

Fig. 3: Eat predatory fish and sharks

Prey predator model (Volterra): the number of prey (A) is x (t), the number of predator (B) is y (t), the growth rate of independent survival is r, then there is $\dot{x} = rx$. B makes the growth rate of A decrease, and a small amount is proportional to y, then there is $\dot{x}(t) = (r - ay)x = rx - axy$. The mortality of independent survival B is *d*, and there is $\dot{y} = -dy$. A makes the mortality of B decrease, and a small amount is proportional to x, then there is $\dot{y}(t) = -(d - bx)y = -dy + bxy$. Among them, a represents the predator capturing ability, and b represents the ability to feed a predator [17].

Stability analysis of Volterra model: $A = \begin{bmatrix} r - ax & -ax \\ by & -d + bx \end{bmatrix}$.

Numerical solution of differential equations can be obtained by using mathematical software MATLAB:

Table 3
Differential equation numerical solution

t	X(t)	Y(t)
0	20.0000	4.0000
0.1000	21.2406	3.9651
0.2000	22.5649	3.9405
...	...	...
5.1000	9.6162	16.7235
5.2000	9.0173	16.2064
...	...	...
9.6000	19.6136	3.9968
9.7000	20.8311	3.9587

In predator-prey model, the number of predators can be expressed in $\bar{y} = \frac{r}{a}$, where r is he growth rate of the prey, and a is the ability to grab prey of the predator. The number of predators is proportional to r and inversely proportional to a.

$\bar{x} = \frac{d}{b}$ represents the number of prey, where *d* is the predator mortality rate, and *b* is the ability to feed on prey. The number of prey is proportional to *d*, and is inversely proportional to *b*.

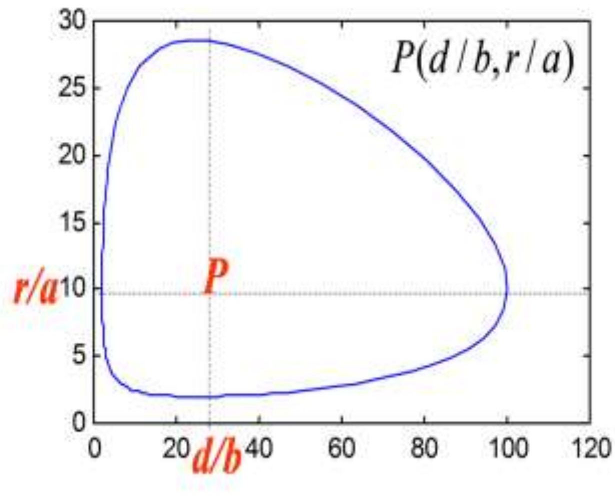

Fig. 4: The prey-predator system model of the models

During World War I, the amount of fishing in the Mediterranean dropped, but the proportion of sharks was increasing, and why? Natural environment model: $P(\bar{x}, \bar{y})$, and $\bar{x} = d/b, \bar{y} = r/a$. Fishing model:

$r \to r-\varepsilon_1, d \to d+\varepsilon_1$, and it can conclude that $\bar{x}_1 > \bar{x}, \bar{y}_1 < \bar{y}$, $P \to P_1$. War fishing: $r \to r-\varepsilon_2, d \to d+\varepsilon_2, \varepsilon_2 < \varepsilon_1$, and it can be seen that $\bar{x}_2 < \bar{x}_1, \bar{y}_2 > \bar{y}_1$, $P_1 \to P_2$.

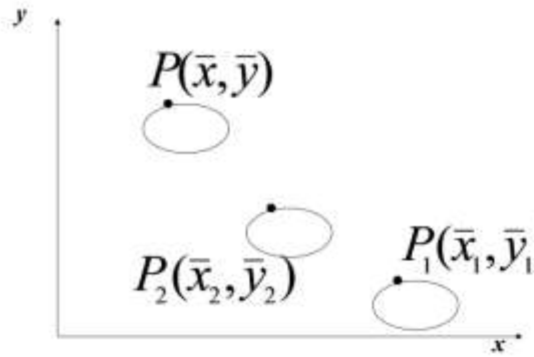

Fig. 5: Continuous decrease, predators increased

$P \to P_1$ also reflects that in the pest (prey) - insects (predator) system, when using two kinds of insecticides destroy insects, pests will increase, and pests will reduce.

Strong coupling mutualism model with Dirichlet boundary conditions

Types of mutualism organisms: A mutually beneficial relationship between the different kinds of two individuals can increase the fitness of both sides. There are several types of mutually beneficial symbiosis. One is only acting in the mutualism, such as shrimp and fish[18].

Fig. 6: Mutualism of shrimp and fish

There is mutually beneficial symbiosis of planting and feeding, such as termites and fungi. Some ants and termites live on the growing fungi. For example, leaf cutting ants (Atta) dig the pit with 2-3 liters in the soil, and cut the leaves in the pit to cultivate fungi from nearby plants. The entire colony of ants is fully grown for their own food on the fungi, while the benefits of fungi from the symbiosis are "feeding" and spreading by the leaf cutter ants. Small beetle digs tunnels in the dead tree stem, and cultivate fungi in suffering for their larvae as food.

There is mutually beneficial symbiosis between flowering plants and pollination animals, such as bees and plants. The benefits of specialized flower in evolution are: The corresponding pollination is that an insect can "know" the flowers, and the honey in the plant between the same species increases the chance of breeding; Reducing the loss of pollen in a plant that is not related to the plant; Through the academic learning and evolution, gathering honey of insects from this special type of flowers is more effective[19].

Fig.7: Flowering plant and animal pollination mutualism

In addition, there are mutual benefits in the animal digestive tract, such as: ruminant animal stomach and ciliate; higher plants and fungi symbiotic; mycorrhizae; and the symbiotic body living in animal tissues or cells, ciliates and algae [20].

Introduction to the strong coupling mutualism model: Strong coupling reciprocity Model with Dirichlet boundary conditions:

$$\begin{cases} -\Delta\left[\left(d_1+a_{11}u_1+\dfrac{a_{12}u_1}{\beta_1+u_2}\right)u_1\right]=r_1u_1\left(1-\dfrac{u_1}{K_1+\alpha_1u_2}\right), x\in\Omega \\ -\Delta\left[\left(d_2+a_{22}u_2+\dfrac{a_{21}u_2}{\beta_2+u_1}\right)u_2\right]=r_2u_2\left(1-\dfrac{u_2}{K_2+\alpha_2u_1}\right), x\in\Omega \\ u_1(x)=0, u_2(x)=0, x\in\partial\Omega \end{cases} \quad (14)$$

Among them, Δ is the Laplasse operator, Ω represents the bounded region of RN, and C $\partial\Omega$ indicates that the boundary is smooth, $u_i\,(i=1,2)$ represents the distribution density of the population. d_1 represents the diffusion rate of u_1, a_{ii} means the self-diffusion coefficient, and a_{ij} represents the cross diffusion coefficient. $K_1+\alpha_1 u_2$ represents the capacity of u_1 population, $K_2+\alpha_2 u_1$ represents the capacity of u_2 population, and $r_i, K_i, a_i\,(i=1,2)$ are positive. The boundary conditions show that the surrounding environment is very bad. If when $x\in\Omega$, there is $u_1(x)>0, u_2(x)>0,$ then (u_1,u_2) is called the positive solution to the system, which is known the coexistence solution.

The $\lambda_0>0$ represents the first eigenvalue of $(-\Delta)$ operator under the Dirichlet boundary condition, $\phi>0$ is the corresponding characteristic function, $\|\phi\|=1$. In this chapter, we first explain the coexistence of strong coupling systems by a general problem, and then give the conditions for the coexistence of the specific model as formula (14). In this model, the first population of J_1 and the flow rate of the second populations of J_2 are:

$$J_1=-\left(d_1+2a_{11}u_1+\frac{2a_{12}u_1}{\beta_1+u_2}\right)\nabla u_1+\frac{a_{12}u_1^2}{\left(\beta_1+u_2\right)^2}\nabla u_2,$$

$$J_2=-\left(d_2+2a_{22}u_2+\frac{2a_{21}u_2}{\beta_2+u_1}\right)\nabla u_2+\frac{a_{21}u_2^2}{\left(\beta_2+u_1\right)^2}\nabla u_1$$

$-\left(d_1+2a_{11}u_1+\frac{2a_{12}u_1}{\beta_1+u_2}\right)$ and $-\left(d_2+2a_{22}u_2+\frac{2a_{21}u_2}{\beta_2+u_1}\right)$ represent the self-diffusion system, $\frac{a_{12}u_1^2}{\left(\beta_1+u_2\right)^2}$ and $\frac{a_{21}u_2^2}{\left(\beta_2+u_1\right)^2}$ represent the cross diffusion system. The $\frac{a_{12}u_1^2}{\left(\beta_1+u_2\right)^2}\nabla u_2$ part of the above shows that the population of u_1 is moving toward the direction of population u_2 with bigger distribution density. Similarly, the second population of u_2 is moving in the direction of population u_1 with bigger distribution density, that is, the two populations are close to each other.

Strong coupling coexistence of mutualism population models: The strong coupling elliptic problem with Dirichlet boundary condition is known in the front. Here we study a specific two-species mutualism model. First, we give the equivalent form of the problem (14):

$$\begin{cases}-\Delta\left[D_1(u_1,u_2)\right]=f_1(u_1,u_2), x\in\Omega\\ -\Delta\left[\left[D_2(u_1,u_2)\right]\right]=f_2(u_1,u_2), x\in\Omega\\ u_1(x)=0, u_2(x)=0, x\in\partial\Omega\end{cases}\tag{15}$$

And then to define $\omega_1=D_1(u_1,u_2)$ and $\omega_2=D_2(u_1,u_2)$; therefore, for $\forall(u_1,u_2)\ge(0,0)$, the Jacobian matrix related to ω_1,ω_2 is:

$$J=\frac{\partial(\omega_1,\omega_2)}{\partial(u_1,u_2)}=\begin{vmatrix}\frac{\partial\omega_1}{\partial u_1} & \frac{\partial\omega_1}{\partial u_2}\\ \frac{\partial\omega_2}{\partial u_1} & \frac{\partial\omega_2}{\partial u_2}\end{vmatrix}=$$

$$\left(d_1+2a_{11}u_1+\frac{2a_{12}u_1}{\beta_1+u_2}\right)\left(d_2+\frac{2a_{12}u_2}{\beta_2+u_1}+2a_{22}u_2\right)-\left[\frac{-a_{12}u_1^2}{\left(\beta_1+u_2\right)^2}\cdot\frac{-a_{21}u_2^2}{\left(\beta_2+u_1\right)^2}\right]\ge d_1d_2>0$$

When $\forall(u_1,u_2)\ge(0,0)$, the inverse function $u_1=g_1(\omega_1,\omega_2), u_2=g_2(\omega_1,\omega_2)$ exists, the strong coupling problem (14) is transformed into equivalent and weak coupling problem:

$$\begin{cases}L_i\omega_i=F_i(u_1,u_2), x\in\Omega\\ u_i=g_i(\omega_1,\omega_2), x\in\Omega\\ \omega_i(x)=0, x\in\partial\Omega\end{cases}\tag{16}$$

Among them, $L_i\omega_i=-\Delta\omega_i+k_i\omega_i, F_i(u_1,u_2)=f_i(u_1,u_2)+k_iD_i(u_1,u_2), i=1,2, k_i$ are not certain. The upper and lower solutions to the weakly coupled system (16) are defined as follows:

Definition 1:

If $(\tilde{u},\tilde{w})=(\tilde{u}_1,\tilde{u}_2,\tilde{\omega}_1,\tilde{\omega}_2)$ and $(\hat{u},\hat{w})=(\hat{u}_1,\hat{u}_2,\hat{\omega}_1,\hat{\omega}_2)$, the vector functions $(\tilde{u},\tilde{w})\in C^2(\Omega)\cap C(\bar{\Omega})$ and $(\hat{u},\hat{w})\in C^2(\Omega)\cap C(\bar{\Omega})$ are the upper and lower solutions to the problem. If $(\tilde{u},\tilde{w})\ge(\hat{u},\hat{w})$, and the component meets the following relationships:

$$\begin{cases}L_i\tilde{\omega}_i\ge F_i(x,u)\ge L_i\tilde{\omega}_i, \hat{u}\le u\le\tilde{u}, x\in\Omega\\ \tilde{u}\ge g_i(x,w)\ge\hat{u}_i, \hat{w}\le w\le\tilde{w}, x\in\Omega\\ \tilde{\omega}_i(x)\ge 0\ge\hat{\omega}_i(x),, x\in\partial\Omega\end{cases}\tag{17}$$

Then the following gives a more concise form equivalent to the problem (4). First of all, to calculate the value of $\frac{\partial u_1}{\partial \omega_1}(i, j = 1, 2)$ to just the monotonic properties of u_1, u_2 to ω_1, ω_2. Though the direct calculation, it can be known that for arbitrary $(u_1, u_2) \geq (0, 0)$,

$$\frac{\partial u_1}{\partial \omega_1} = \frac{\partial \omega_2}{\partial u_2} J^{-1} = \left(d_2 + \frac{2a_{21}}{\beta_2 + u_1} + 2a_{22} \right) J^{-1} > 0,$$

$$\frac{\partial u_2}{\partial \omega_1} = -\frac{\partial \omega_2}{\partial u_1} J^{-1} = \frac{a_{21} u_2^2}{(\beta_2 + u_1)^2} J^{-1} > 0,$$

$$\frac{\partial u_2}{\partial \omega_2} = \frac{\partial \omega_1}{\partial u_1} J^{-1} = \left(d_2 + \frac{2a_{12}}{\beta_1 + u_2} + 2a_{11} \right) J^{-1} > 0,$$

$$\frac{\partial u_1}{\partial \omega_2} = -\frac{\partial \omega_1}{\partial u_2} J^{-1} = \frac{a_{12} u_1^2}{(\beta_1 + u_2)^2} J^{-1} > 0.$$

Therefore, for arbitrary $(\omega_1, \omega_2) \geq (0, 0)$, $u_1 = g_1(\omega_1, \omega_2)$ and $u_2 = g_2(\omega_1, \omega_2)$ are monotone and non-reduced for ω_1, ω_2, noted $\tilde{u}_1 = g_1(\tilde{\omega}_1, \tilde{\omega}_2)$, $\tilde{u}_2 = g_2(\tilde{\omega}_1, \tilde{\omega}_2)$, $\hat{u}_1 = g_1(\hat{\omega}_1, \hat{\omega}_2)$, $\hat{u}_2 = g_2(\hat{\omega}_1, \hat{\omega}_2)$.

The above relationship is equivalent to:

$$\tilde{\omega}_1 = D_1(\tilde{u}_1, \tilde{u}_2), \tilde{\omega}_2 = D_2(\tilde{u}_1, \tilde{u}_2),$$

$$\hat{\omega}_1 = D_1(\hat{u}_1, \hat{u}_2), \hat{\omega}_2 = D_2(\hat{u}_1, \hat{u}_2).$$

Definition 2:

If $(\tilde{u}, \tilde{w}) = (\tilde{u}_1, \tilde{u}_2, \tilde{\omega}_1, \tilde{\omega}_2)$ and $(\hat{u}, \hat{w}) = (\hat{u}_1, \hat{u}_2, \hat{\omega}_1, \hat{\omega}_2)$, the vector functions $(\tilde{u}, \tilde{w}) \in C^2(\Omega) \cap C(\bar{\Omega})$ and $(\hat{u}, \hat{w}) \in C^2(\Omega) \cap C(\bar{\Omega})$ are the upper and lower solutions to the problem. If $(\tilde{u}, \tilde{w}) \geq (\hat{u}, \hat{w})$ and the component meets the following relationships:

$$\begin{cases} -\Delta\left(D_1(\tilde{u}_1, \tilde{u}_2)\right) + k_1 D_1(\tilde{u}_1, \tilde{u}_2) \geq F_1(\tilde{u}_1, \tilde{u}_2), x \in \Omega \\ -\Delta\left(D_2(\tilde{u}_1, \tilde{u}_2)\right) + k_2 D_2(\tilde{u}_1, \tilde{u}_2) \geq F_2(\tilde{u}_1, \tilde{u}_2), x \in \Omega \\ -\Delta\left(D_1(\hat{u}_1, \hat{u}_2)\right) + k_1 D_1(\hat{u}_1, \hat{u}_2) \geq F_1(\hat{u}_1, \hat{u}_2), x \in \Omega \\ -\Delta\left(D_2(\hat{u}_1, \hat{u}_2)\right) + k_2 D_2(\hat{u}_1, \hat{u}_2) \geq F_2(\hat{u}_1, \hat{u}_2), x \in \Omega \\ \tilde{u}_i(x) \geq 0 \geq \hat{u}_i(x), i = 1, 2, x \in \partial\Omega, \end{cases} \quad (18)$$

Then

$$(\tilde{u}_1, \tilde{u}_2) = (M_1, M_2), (\hat{u}_1, \hat{u}_2) = \left(g_1(\delta_1\phi, \delta_2\phi), g_2(\delta_1\phi, \delta_2\phi)\right),$$

M_i and δ_i are constant, and $\delta_i (i = 1, 2)$ is small enough. The positive characteristic function of λ_0 satisfies $-\Delta\phi(x) = \lambda_0 \phi(x)$. The structure of the lower solution is guaranteed:

$$\delta_1\phi = \left(d_1 + a_{11}\hat{u}_1 + \frac{a_{12}\hat{u}_1}{\beta_2 + \hat{u}_2} \right) \hat{u}_1,$$

$$\delta_2\phi = \left(d_2 + a_{22}\hat{u}_2 + \frac{a_{21}\hat{u}_2}{\beta_2 + \hat{u}_2} \right) \hat{u}_2 \quad (19)$$

So if (M_1, M_2), $g_1(\delta_1\phi, \delta_2\phi)$ and $g_2(\delta_1\phi, \delta_2\phi)$ can satisfy the following relationships:

$$\begin{cases} -\Delta\left(\left(d_1 + a_{11}M_1 + \frac{a_{12}M_1}{\beta_1 + M_2}\right) M_1\right) \geq r_1 M_1 \left(1 - \frac{M_1}{K_1 + \alpha_1 M_2}\right), \\ -\Delta\left(\left(d_2 + a_{22}M_2 + \frac{a_{21}M_2}{\beta_2 + M_1}\right) M_2\right) \geq r_2 M_2 \left(1 - \frac{M_2}{K_2 + \alpha_2 M_1}\right), \end{cases} \quad (20)$$

$$\begin{cases} \lambda_0 \delta_1 \phi \leq r_1 \delta_1 \phi \left(1 - \frac{\hat{u}_1}{K_1 + \alpha_1 \hat{u}_2}\right) \left(d_1 + a_{11}\hat{u}_1 + \frac{a_{12}\hat{u}_1}{\beta_2 + \hat{u}_2}\right)^{-1} \\ \lambda_0 \delta_2 \phi \leq r_2 \delta_2 \phi \left(1 - \frac{\hat{u}_2}{K_2 + \alpha_2 \hat{u}_1}\right) \left(d_2 + a_{22}\hat{u}_2 + \frac{a_{21}\hat{u}_2}{\beta_2 + \hat{u}_1}\right)^{-1} \end{cases} \quad (21)$$

Then the condition (18) can be established.

First of all, the condition (20) is tested. Assuming that $a_1 a_2 < 1$, as long as $a_1 a_2 < 1$ and

$$M_1 = L\frac{K_1 + \alpha_1 K_2}{1 - \alpha_1 \alpha_2}, M_2 = L\frac{K_2 + \alpha_2 K_1}{1 - \alpha_1 \alpha_2} (L \geq 1) \quad (22)$$

Then the condition (20) is satisfied. Look at the condition (21), with the assumptions:

$$\lambda_0 d_1 < r_1, \lambda_0 d_2 < r_2 \quad (23)$$

As long as δ_1 and δ_2 are small enough, the condition (9) can be satisfied. Thus, under the conditions (22) and (23), we can find the normal number $M_i, \delta_i (i = 1, 2)$, so that $(\tilde{u}_1, \tilde{u}_2), (\hat{u}_1, \hat{u}_2)$ satisfy the order upper and lower solutions to the problem (14) defined in (19). Though the study, theorem 1 can be concluded, assuming the conditions (22) and (23) are established, the problem (14) has at least one positive solution (u_1, u_2), which can satisfy $(\tilde{u}_1, \tilde{u}_2) \geq (u_1, u_2) \geq (\hat{u}_1, \hat{u}_2)$, and $u = (u_1, u_2)$ is a solution of the problem. When $\lambda_0 d_1 < r_1, \lambda_0 d_2 < r_2$, a_{12} and a_{21} are small enough, that is, when the interaction between diffusion and population is relatively weak, the problem (15) has at least one coexistence solution.

Conclusions

With the development of science and technology, the deepening integration, mutual cooperation and penetration between the various disciplines, and cross discipline between activities will be the trends of research development in the future, which are also important ways to produce innovative achievements. Biological mathematics has become one of the hot spots in the research of modern applied mathematics. The biological population model is established in this paper, and the process change trends with a long time is studied from the aspects of the competition model, interdependence model, and the model of the law of the jungle. Based on the local stability, the phase trajectory is analyzed to obtain the global stability of equilibrium point. The knowledge of biological mathematics is studied, and examples of a kind of reciprocity model are given. On the basis of these examples, the coexistence is studied to obtain the existence conclusion of strong coupling of mutualism population models: When $\lambda_0 d_1 < r_1, \lambda_0 d_2 < r_2$, a_{12} and a_{21} are small enough, that is, when the interaction between diffusion and population is relatively weak, the problem (15) has at least one coexistence solution. The coexistence of competition and predation model is not considered in this study, and there are some limitations which can be further in-depth studied in the latter research.

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